

Not all physical errors can be linear CPTP maps in a correlation space

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In the framework of quantum computational tensor network, which is a general framework of measurement-based quantum computation, the resource many-body state is represented in a tensor-network form, and universal quantum computation is performed in a virtual linear space, which is called a correlation space, where tensors live. Since any unitary operation, state preparation, and the projection measurement in the computational basis can be simulated in a correlation space, it is natural to expect that fault-tolerant quantum circuits can also be simulated in a correlation space. However, we point out that not all physical errors on physical qudits appear as linear completely-positive trace-preserving errors in a correlation space. Since the theories of fault-tolerant quantum circuits known so far assume such noises, this means that the simulation of fault-tolerant quantum circuits in a correlation space is not so straightforward for general resource states.

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Introduction.— Quantum many-body states, which have long been central research objects in condensed matter physics, statistical physics, and quantum chemistry, are now attracting the renewed interest in quantum information science as fundamental resources for quantum information processing. One of the most celebrated examples is one-way quantum computation [1–3]. Once the highly-entangled many-body state which is called the cluster state is prepared, universal quantum computation is possible with adaptive local measurements on each qubit. Recently, the concept of quantum computational tensor network (QCTN) [4–6], which is the general framework of measurement-based quantum computation on quantum many-body states, was proposed. This novel framework has enabled us to understand how general measurement-based quantum computation is performed on many other resource states beyond the cluster state. The most innovative feature of QCTN is that the resource state is represented in a tensor network form [7–9], and universal quantum computation is performed in the virtual linear space where tensors live. For example, let us consider the one-dimensional open-boundary chain of N qudits in the matrix product form

$$|\Psi\rangle \equiv \sum_{k_1=0}^{d-1} \dots \sum_{k_N=0}^{d-1} \langle L|A[k_N] \dots A[k_1]|R\rangle |k_N, \dots, k_1\rangle, \quad (1)$$

where $\{|0\rangle, \dots, |d-1\rangle\}$ is a certain basis in the d -dimensional Hilbert space ($2 \leq d < \infty$), $|L\rangle$ and $|R\rangle$ are D -dimensional complex vectors, and $\{A[0], \dots, A[d-1]\}$ are $D \times D$ complex matrices. Let us also define the projection measurement $\mathcal{M}_{\theta, \phi}$ on a single physical qudit by $\mathcal{M}_{\theta, \phi} \equiv \{|\alpha_{\theta, \phi}\rangle, |\beta_{\theta, \phi}\rangle, |2\rangle, \dots, |d-1\rangle\}$, where $|\alpha_{\theta, \phi}\rangle \equiv \cos \frac{\theta}{2}|0\rangle + e^{i\phi} \sin \frac{\theta}{2}|1\rangle$, $|\beta_{\theta, \phi}\rangle \equiv \sin \frac{\theta}{2}|0\rangle - e^{i\phi} \cos \frac{\theta}{2}|1\rangle$, $0 < \theta < \pi$, and $0 \leq \phi < 2\pi$. If we do the measurement $\mathcal{M}_{\theta, \phi}$ on the first physical qudit of Eq. (1) and if the first physical qudit is projected onto, for example, $|\alpha_{\theta, \phi}\rangle$ as a result of this measurement, the state Eq. (1)

becomes

$$\sum_{k_2, \dots, k_N} \langle L|A[k_N] \dots A[k_2] \frac{A[\alpha_{\theta, \phi}]}{\|A[\alpha_{\theta, \phi}]\|} |R\rangle |k_N, \dots, k_2\rangle \otimes |\alpha_{\theta, \phi}\rangle,$$

where $A[\alpha_{\theta, \phi}] \equiv \cos \frac{\theta}{2} A[0] + e^{-i\phi} \sin \frac{\theta}{2} A[1]$. Then, we say “the operation $|R\rangle \rightarrow \frac{A[\alpha_{\theta, \phi}]}{\|A[\alpha_{\theta, \phi}]\|} |R\rangle$ is implemented in the correlation space”. In particular, if $A[0]$, $A[1]$, θ , and ϕ are appropriately chosen in such a way that $A[\alpha_{\theta, \phi}]$ is proportional to a unitary, we can “simulate” the unitary evolution $\frac{A[\alpha_{\theta, \phi}]}{\|A[\alpha_{\theta, \phi}]\|} |R\rangle$ of the vector $|R\rangle$ in the virtual linear space where A ’s, $|R\rangle$, and $|L\rangle$ live. This virtual linear space is called the correlation space [4–6]. The core of QCTN is this “virtual quantum computation” in the correlation space. If the correlation space has a sufficient structure and if A ’s, $|L\rangle$, and $|R\rangle$ are appropriately chosen, we can “simulate” universal quantum circuit in the correlation space [4–6, 10, 11].

For the realization of a scalable quantum computer, a theory of fault-tolerant (FT) quantum computation [12–16] is necessary. In fact, several researches have been performed on FT quantum computation in the one-way model [3, 17–21]. However, there has been no result about a theory of FT quantum computation on general QCTN [22]. In particular, there is severe lack of knowledge about FT quantum computation on resource states with $d \geq 3$. It is necessary to consider resource states with $d \geq 3$ if we want to enjoy the cooling preparation of a resource state and the energy-gap protection of measurement-based quantum computation with a physically natural Hamiltonian, since no genuinely entangled qubit state can be the unique ground state of a two-body frustration-free Hamiltonian [26].

One straightforward way of implementing FT quantum computation on QCTN is to encode physical qudits with a quantum error correcting code: $\sum_{k_1=0}^{d-1} \dots \sum_{k_N=0}^{d-1} \langle L|A[k_N] \dots A[k_1]|R\rangle |\hat{k}_N, \dots, \hat{k}_1\rangle$, where

$|\tilde{k}_i\rangle$ ($i = 1, \dots, N$) is the encoded version of $|k_i\rangle$ (such as $|\tilde{0}\rangle = |000\rangle$ and $|\tilde{1}\rangle = |111\rangle$, etc.) In fact, this strategy was taken in Refs. [20, 21] for the one-way model ($d = 2$), and it was shown there that a FT construction of the encoded cluster state is possible. For $d \geq 3$, however, such a strategy is difficult, since theories of quantum error correcting codes and FT preparations of the encoded resource state are less developed for $d \geq 3$. Furthermore, if we encode physical qudits with a quantum error correcting code, the parent Hamiltonian should no longer be two-body interacting one.

The other way of implementing FT quantum computation on QCTN is to simulate FT quantum circuits in the correlation space. Since any unitary operation, state preparation, and the projective measurement in the computational basis can be simulated in a correlation space (for a precise discussion about the possibility of the measurement, see Ref. [11]), it is natural to expect that FT quantum circuits can also be simulated in a correlation space. An advantage of this strategy is that theories of FT quantum circuits for qubit systems are well developed [12–16]. In fact, this strategy was taken in Refs. [3, 17, 18] for the one-way model ($d = 2$). They simulated FT quantum circuits on the one-way model.

In this paper, however, we point out that it is not so straightforward to simulate FT quantum circuits in a correlation space in general. We first consider the simulation of quantum circuits in the correlation space of pure matrix product states (post-measurement conditional states). We show that if $d \geq 3$ not all physical errors on physical qudits appear as linear completely-positive trace-preserving (CPTP) errors in the correlation space. Since all theories of FT quantum circuits known so far assume such noises [12–16], this means that it is not so straightforward to apply these FT theories to quantum circuits simulated in the correlation space of pure matrix product states.

We therefore next consider another way of simulating quantum circuits in the correlation space by mixing measurement results. For the cluster state and the tricluster state [27], such a mixing strategy well works: all CPTP errors on a physical qubit (or qudit) can be CPTP errors in the correlation space. However, this is not the case for other general resource states. As an example, we consider the one-dimensional AKLT state [28, 29], and see that not all physical errors on a physical qudit can be linear CPTP errors in the correlation space even if we mix measurement results. This suggests that even if we mix measurement results, like the cluster model, the simulation of FT quantum circuits in the correlation space of a general resource state is not so straightforward.

Simulation with pure states.— First, let us consider the simulation of quantum circuits in the correlation space of pure matrix product states. We show that if $d \geq 3$ not all physical errors on physical qudits appear as linear CPTP errors in the correlation space.

Since the MPS, Eq. (1), is a resource state for measurement-based quantum computation, we can assume without loss of generality that $A[\alpha_{\theta,\phi}]$, $A[\beta_{\theta,\phi}]$, $A[2]$, $A[3]$, ..., $A[d-1]$ are unitary up to constants: $A[\alpha_{\theta,\phi}] = c_\alpha U_\alpha$, $A[\beta_{\theta,\phi}] = c_\beta U_\beta$, $A[2] = c_2 U_2$, $A[3] = c_3 U_3$, ..., $A[d-1] = c_{d-1} U_{d-1}$, where c_α , c_β , c_2 , ..., c_{d-1} are real positive numbers, U_α , U_β , U_2 , ..., U_{d-1} are unitary operators, and $A[\beta_{\theta,\phi}] \equiv \sin \frac{\theta}{2} A[0] - e^{-i\phi} \cos \frac{\theta}{2} A[1]$. This means that any operation implemented in the correlation space by the measurement $\mathcal{M}_{\theta,\phi}$ on a single physical qudit of Eq. (1) is unitary. Note that this assumption is reasonable, since otherwise Eq. (1) does not seem to be useful as a resource for measurement-based quantum computation. In fact, all known resource states so far [1–6, 27, 28, 30, 31], including the cluster state and the AKLT state, satisfy this assumption by appropriately rotating each local physical basis. Furthermore, we can take c_α , c_β , c_2 , ..., c_{d-1} such that $C \equiv c_\alpha^2 + c_\beta^2 + \sum_{k=2}^{d-1} c_k^2 = 1$, since $\sum_{k_1, \dots, k_N} \langle L[A[k_N]] \dots A[k_1] | R \rangle | k_N, \dots, k_1 \rangle = \sqrt{C}^N \sum_{k_1, \dots, k_N} \langle L[\frac{A[k_N]}{\sqrt{C}}] \dots \frac{A[k_1]}{\sqrt{C}} | R \rangle | k_N, \dots, k_1 \rangle$ and we can redefine $A[k_i]/\sqrt{C} \rightarrow A[k_i]$.

Theorem: If $d \geq 3$, there exists a single-qudit CPTP error \mathcal{E} which has the following property: assume that \mathcal{E} is applied on a single physical qudit of Eq. (1). If the measurement $\mathcal{M}_{\theta,\phi}$ is performed on that affected qudit, a non-TP operation is implemented in the correlation space.

Proof: In order to show Theorem, let us assume that

$$\text{There is no such } \mathcal{E}. \quad (2)$$

We will see that this assumption leads to the contradiction that $d \leq 2$.

First, let us consider the state

$$(I^{\otimes N-1} \otimes U_{1 \leftrightarrow 2}) |\Psi\rangle, \quad (3)$$

where $U_{a \leftrightarrow b} \equiv |a\rangle\langle b| + |b\rangle\langle a| + I - |a\rangle\langle a| - |b\rangle\langle b|$ is the unitary error which exchanges $|a\rangle$ and $|b\rangle$, and I is the identity operator on a single qudit. In Eq. (3), the error $U_{1 \leftrightarrow 2}$ is applied on the first physical qudit of $|\Psi\rangle$. If we do the measurement $\mathcal{M}_{\theta,\phi}$ on the first physical qudit of Eq. (3), and if the measurement result is $|2\rangle$, Eq. (3) becomes

$$\sum_{k_2, \dots, k_N} \langle L[A[k_N]] \dots A[k_2] \frac{A[1]}{\|A[1]\|} | R \rangle | k_N, \dots, k_2 \rangle \otimes |2\rangle. \quad (4)$$

In other words, the operation $|R\rangle \rightarrow \frac{A[1]}{\|A[1]\|} |R\rangle$ is implemented in the correlation space. By the assumption Eq. (2), this operation should work as a TP operation in the correlation space. Therefore,

$$\frac{A^\dagger[1]}{\|A[1]\|} \frac{A[1]}{\|A[1]\|} = I. \quad (5)$$

By taking $\eta \equiv \|A[1]\|^2$, we obtain

$$A^\dagger[1]A[1] = \eta I. \quad (6)$$

Second, let us consider the measurement $\mathcal{M}_{\theta,\phi}$ on the first physical qudit of $(I^{\otimes N-1} \otimes U_{0 \leftrightarrow 2} V^s)|\Psi\rangle$, where $s \in \{0, 1, \dots, d-1\}$, $V \equiv \sum_{p=0}^{d-1} e^{-i\omega p}|p\rangle\langle p|$ is a unitary phase error, and $\omega \equiv 2\pi/d$. If the measurement result is $|\alpha_{\theta,\phi}\rangle$, $(e^{-2is\omega} \cos \frac{\theta}{2} A[2] + e^{-i(\phi+s\omega)} \sin \frac{\theta}{2} A[1])/\sqrt{\gamma}$ is implemented in the correlation space, where $\sqrt{\gamma} \equiv \|e^{-2is\omega} \cos \frac{\theta}{2} A[2] + e^{-i(\phi+s\omega)} \sin \frac{\theta}{2} A[1]\|$. By the assumption Eq. (2), this should work as a TP operation in the correlation space. Therefore, $\gamma I = \cos^2 \frac{\theta}{2} A^\dagger[2]A[2] + \sin^2 \frac{\theta}{2} A^\dagger[1]A[1] + \frac{1}{2} \sin \theta (e^{-i(\phi-s\omega)} A^\dagger[2]A[1] + e^{i(\phi-s\omega)} A^\dagger[1]A[2])$. By the assumption that all A 's are proportional to unitaries, $A^\dagger[2]A[2] = \xi I$, where $\xi \equiv \|A[2]\|^2$. Furthermore, as we have shown, $A^\dagger[1]A[1] = \eta I$ (Eq. (6)). Therefore,

$$\gamma' I = e^{-i(\phi-s\omega)} A^\dagger[2]A[1] + e^{i(\phi-s\omega)} A^\dagger[1]A[2], \quad (7)$$

where $\gamma' \equiv \frac{2}{\sin \theta} (\gamma - \xi \cos^2 \frac{\theta}{2} - \eta \sin^2 \frac{\theta}{2})$.

Finally, let us consider the measurement $\mathcal{M}_{\theta,\phi}$ on the first physical qudit of $(I^{\otimes N-1} \otimes U_{0 \leftrightarrow 1} U_{0 \leftrightarrow 2} V^t)|\Psi\rangle$, where $t \in \{0, 1, \dots, d-1\}$. If the measurement result is $|\alpha_{\theta,\phi}\rangle$, $(e^{-it\omega} \cos \frac{\theta}{2} A[1] + e^{-i(\phi-2it\omega)} \sin \frac{\theta}{2} A[2])/\sqrt{\delta}$ is implemented in the correlation space, where $\sqrt{\delta} \equiv \|e^{-it\omega} \cos \frac{\theta}{2} A[1] + e^{-i(\phi-2it\omega)} \sin \frac{\theta}{2} A[2]\|$. By the assumption Eq. (2), this should also work as a TP operation in the correlation space. Therefore,

$$\delta' I = e^{i(\phi+t\omega)} A^\dagger[2]A[1] + e^{-i(\phi+t\omega)} A^\dagger[1]A[2], \quad (8)$$

where $\delta' \equiv \frac{2}{\sin \theta} (\delta - \xi \sin^2 \frac{\theta}{2} - \eta \cos^2 \frac{\theta}{2})$.

From Eqs. (7) and (8), $\epsilon I = [e^{-2i(\phi-s\omega)} - e^{2i(\phi+t\omega)}] A^\dagger[2]A[1]$, where $\epsilon \equiv e^{-i(\phi-s\omega)} \gamma' - e^{i(\phi+t\omega)} \delta'$.

Let us assume that $e^{-2i(\phi-s\omega)} - e^{2i(\phi+t\omega)} \neq 0$. Then, $\epsilon' I = A^\dagger[2]A[1]$, where $\epsilon' \equiv \epsilon / (e^{-2i(\phi-s\omega)} - e^{2i(\phi+t\omega)})$. If $\epsilon' = 0$, $A^\dagger[2]A[1] = 0$, which means $A[1] = 0$ since $A[2]$ is unitary up to a constant. Therefore, $\epsilon' \neq 0$. In this case, $A[1] = \epsilon'' A[2]$ for certain $\epsilon'' \neq 0$, since $A[2]$ is unitary up to a constant [32]. Hence $e^{-2i(\phi-s\omega)} - e^{2i(\phi+t\omega)} = 0$. This means

$$2\phi + (t-s)\omega = r_{s,t}\pi, \quad (9)$$

where $r_{s,t} \in \{0, 1, 2, 3, \dots\}$. Let us take $t = s = 0$. Then, Eq. (9) gives $\phi = r_{0,0} \frac{\pi}{2}$ ($r_{0,0} \in \{0, 1, 2, \dots\}$). Let us take $s = 1, t = 0$. Then, Eq. (9) gives $\phi = \frac{\pi}{d} + r_{1,0} \frac{\pi}{2}$ ($r_{1,0} \in \{0, 1, 2, \dots\}$). In order to satisfy these two equations at the same time, there must exist $r_{0,0}$ and $r_{1,0}$ such that $r_{0,0} \frac{\pi}{2} = \frac{\pi}{d} + r_{1,0} \frac{\pi}{2}$. If $r_{0,0} = r_{1,0}$, then $0 = 1/d$ which means $d = \infty$. Therefore $r_{0,0} \neq r_{1,0}$. Then we have $d = 2/(r_{0,0} - r_{1,0}) \leq 2$, which is the contradiction. ■

One might think that if we rewrite the post-measurement state Eq. (4) as

$$\sum_{k_2, \dots, k_N} \langle L|A[k_N] \dots A[k_2] \frac{A[1]}{\|A[1]\|} |R\rangle |k_N, \dots, K_2\rangle \otimes |2\rangle$$

and redefine the operation implemented in the correlation space as $|R\rangle \rightarrow \frac{A[1]}{\|A[1]\|} |R\rangle$, the TP-ness is recovered in

the correlation space. However, in this case, the non-linearity appears unless $A^\dagger[1]A[1] \propto I$, and therefore if we require the linearity in the correlation space, we obtain the same contradiction.

In short, if $d \geq 3$ not all physical errors on physical qudits appear as linear CPTP errors in the correlation space of pure matrix product states [33].

Simulation by mixing measurement results.— We have seen that if we simulate quantum circuits in the correlation space of pure matrix product states, not all physical errors on a physical qudit can be linear CPTP errors in the correlation space. Therefore we must simulate quantum circuits in the correlation space with another method: we consider the simulation by mixing measurement results.

Before studying a concrete example, let us consider the effect of a CPTP error on general resource states. Let us assume that a CPTP error $\rho \rightarrow \sum_{j=1}^w F_j \rho F_j^\dagger$, where $\sum_{j=1}^w F_j^\dagger F_j = I$, occurs on the first physical qudit of $|\Psi\rangle$. If we measure the first physical qudit in a certain basis $\{|m_s\rangle\}$, we obtain [34] $\sum_s \sum_j W(E_{j,s}|R)_2 \otimes |m_s\rangle\langle m_s|$, where $E_{j,s} \equiv \sum_k A[k] \langle m_s | F_j | k \rangle$ and

$$W(|\psi\rangle)_r \equiv \sum_{k_r, \dots, k_N} \sum_{k'_r, \dots, k'_N} \langle L|A[k_N] \dots A[k_r] |\psi\rangle \langle \psi | A^\dagger[k'_r] \dots A^\dagger[k'_N] |L\rangle |k_r, \dots, k_N\rangle \langle k'_r, \dots, k'_N|.$$

If we trace out $|m_s\rangle$, we obtain $\sum_s \sum_j W(E_{j,s}|R)_2$. This means that the map

$$|R\rangle\langle R| \rightarrow \sum_{s,j} E_{j,s} |R\rangle\langle R| E_{j,s}^\dagger \quad (10)$$

is implemented in the correlation space. Since we can show [34] $\sum_{j,s} E_{j,s}^\dagger E_{j,s} = I$, the map Eq. (10) is CPTP.

In general, we must do feed-forwarding before tracing out $|m_s\rangle$ in order to deterministically implement quantum gates in the correlation space. As is shown in Refs. [18, 34], all CPTP errors on a physical qubit (or qudit) can be CPTP errors in the correlation space of the cluster state and the tricluster state even if we do the feed-forwarding in this mixing strategy. However, it is not the case for other general resource states [35]. As an example, let us consider the one-dimensional AKLT state, where $d = 3$, $A[0] = \frac{1}{\sqrt{3}}X$, $A[1] = \frac{1}{\sqrt{3}}XZ$, $A[2] = \frac{1}{\sqrt{3}}Z$.

If a CPTP error occurs on the first physical qudit, and if we do the usual measurement-based quantum computation on the AKLT state, we obtain [34]

$$\sum_j \sum_{p=0}^1 \sum_{q=0}^1 \left(\sum_{(s_1, \dots, s_r) \in S_{p,q}^r} W(Q_r(s_1, \dots, s_r) \dots Q_2(s_1, s_2) E_{j,s_1} |R\rangle)_{r+1} + h(p, q, r) W(Z^{r-1} E_{j,2} |R\rangle)_{r+1} \right) \otimes \eta(p, q), \quad (11)$$

where r is the number of measurements (since the gate operation is non-deterministic in AKLT model, we must repeat measurements until we near-deterministically implement the desired gate operation), and $S_{p,q}^r$ is the set of measurement outcomes $(s_1, \dots, s_r) \in \{0, 1, 2\}^r \setminus (2, \dots, 2)$ such that $f(s_1, \dots, s_r) = p$ and $g(s_1, \dots, s_r) = q$. Here, $f(s_1, \dots, s_r) = \bigoplus_{i=1}^r (\delta_{s_i,0} \oplus \delta_{s_i,1})$ and $g(s_1, \dots, s_r) = \bigoplus_{i=1}^r (\delta_{s_i,1} \oplus \delta_{s_i,2})$. Also, $Q_k(s_1, \dots, s_{k-1}, 0) = X e^{iZ\theta/2}$, $Q_k(s_1, \dots, s_{k-1}, 1) = X Z e^{iZ\theta/2}$, $Q_k(s_1, \dots, s_{k-1}, 2) = Z$ for $s_1 = \dots = s_{k-1} = 2$, and $Q_k(s_1, \dots, s_{k-1}, 0) = X$, $Q_k(s_1, \dots, s_{k-1}, 1) = XZ$, $Q_k(s_1, \dots, s_{k-1}, 2) = Z$ for other s_i 's. We also define $h(p, q, r) = \delta_{p,0} \delta_{q,0}$ if r is even, and $h(p, q, r) = \delta_{p,0} \delta_{q,1}$ if r is odd. Finally, $\eta(0,0)$, $\eta(0,1)$, $\eta(1,0)$, and $\eta(1,1)$ are mutually orthogonal states, which record Pauli byproducts. The first term of Eq. (11) corresponds to the mixture of successful measurement results (desired rotation is implemented) and the second term corresponds to the failed measurement results (the desired rotation is not implemented.)

Equation (11) means that for fixed p and q , the map

$$\begin{aligned} |R\rangle\langle R| \mapsto & \sum_{(s_1, \dots, s_r) \in S_{p,q}^r} \sum_j \tilde{Q}(s_1, \dots, s_r, j) |R\rangle\langle R| \tilde{Q}^\dagger(s_1, \dots, s_r, j) \\ & + h(p, q, r) \sum_j Z^{r-1} E_{j,2} |R\rangle\langle R| E_{j,2}^\dagger Z^{r-1} \end{aligned} \quad (12)$$

is implemented in the correlation space, where $\tilde{Q}(s_1, \dots, s_r, j) \equiv Q_r(s_1, \dots, s_r) \dots Q_2(s_1, s_2) E_{j,s_1}$.

For example, let us consider the error with $w = 1$ and

$$F_1 = U_{\mathcal{M}_{\theta,\phi}} \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0| - \frac{|0\rangle - |1\rangle}{\sqrt{2}} \langle 1| + |2\rangle\langle 2| \right),$$

where $U_{\mathcal{M}_{\theta,\phi}} \equiv |\alpha_{\theta,\phi}\rangle\langle 0| + |\beta_{\theta,\phi}\rangle\langle 1| + \sum_{k=2}^{d-1} |k\rangle\langle k|$. Then, if $p = 1$ and $q = 0$, we can show that [34]

$$\sum_{(s_1, \dots, s_r) \in S_{p,q}^r} \sum_j \tilde{Q}^\dagger(s_1, \dots, s_r, j) \tilde{Q}(s_1, \dots, s_r, j)$$

is $\alpha I + \frac{2}{3}|1\rangle\langle 1|$ if r is odd, and $\beta I + \frac{2}{3}|0\rangle\langle 0|$ if r is even, where α and β are certain positive numbers [34]. This means that the map Eq. (12) is not linear CPTP.

Conclusion.— In this paper, we have studied how physical errors on a physical qudit appear in the correlation space. We have shown that if $d \geq 3$ not all physical errors can be linear CPTP errors in the correlation space of pure matrix product states. We have also shown that even if we mix the measurement results, not all physical errors are linear CPTP errors in the correlation space of general resource states. These results suggest that the application of the theories of fault-tolerant quantum circuits to the correlation space is not so straightforward.

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 - [32] We do not consider the case where $A[1] \propto A[2]$, since in this case we can reduce the dimension d to $d - 1$ by redefining the basis of the two-dimensional subspace spanned by $|1\rangle$ and $|2\rangle$.
 - [33] This result is reasonable since we physically do a non-linear (or non-TP) operation. If we consider this fact, it is surprising that linear CPTP operations are implemented by doing non-linear (or non-TP) physical operations on several resource states such as the cluster state when physical operations are perfect!
 - [34] For details, see T. Morimae and K. Fujii, arXiv: 1110.4182.
 - [35] For the AKLT-like resource state, $A[0] = X$, $A[1] = XZ$, and $A[2] = H$, which was proposed in Ref. [4, 5], it seems to be impossible to consider such a mixing strategy due to the existence of the Hadamard H .